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# Recurrence relations for connection coefficients between two families of orthogonal polynomials

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**Abstract**

We describe a simple approach in order to build recursively the connection coefficients between two families of orthogonal polynomial solutions of second- and fourth-order differential equations.

*Keywords:* Orthogonal polynomials; Connection coefficients

## 1. The problem

Let us consider two families of orthogonal polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$ linked by the so-called [2] connection coefficients $C_m(n)$ given by the expansion of the $P_n$ family in terms of the $Q_n$ basis:

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x). \tag{1}$$

These connection coefficients play an important role in many situations of Pure and Applied Mathematics or in Mathematical Physics, where the nonnegative character of these coefficients has received particular attention [1, 2, 4, 9, 10].

The aim of this letter is to describe an elementary procedure in order to find recurrence relations, sometimes easy to solve, between the coefficients $C_m(n)$ when both $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$

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families belong to the classical class of orthogonal polynomials: Jacobi (J), Bessel (B), Laguerre (L), Hermite (H). This gives an alternate way to be compared to a recent approach [5] restricted to Jacobi expansion. Moreover, extensions to nonclassical polynomials and some examples are given.

Let us write the recurrence relation, the structure relation and the second-order differential equation satisfied by the $P_n$ family in the following way:

$$xP_n = A_n P_{n+1} + B_n P_n + C_n P_{n-1} \quad (n \geqslant 0,\ P_{-1} = 0 \text{ and } P_0 = 1), \tag{2}$$

$$\sigma P_n' = \alpha_n P_{n+1} + \beta_n P_n + \gamma_n P_{n-1} \quad (n \geqslant 0 \text{ and } \alpha_0 = \beta_0 = \gamma_0 = 0), \tag{3}$$

$$L_2[P_n] := \sigma P_n'' + \tau P_n' + \lambda_n P_n = 0, \tag{4}$$

where the notation $P_n := P_n(x)$, $\sigma := \sigma(x)$, $\tau := \tau(x)$ has been used. The coefficients $A_n$, $B_n$, $C_n$, $\alpha_n$, $\beta_n$, $\gamma_n$ are well known in the four classical cases [6]. Moreover, one has (see e.g. [6]): $\sigma_{\mathrm{J}} = 1 - x^2$, $\sigma_{\mathrm{B}} = x^2$, $\sigma_{\mathrm{L}} = x$, $\sigma_{\mathrm{H}} = 1$ and the polynomial of first degree $\tau := \tau(x)$ is given by $(\sigma\rho)'/\rho$ where the weight $\rho := \rho(x)$ is, respectively: $\rho_{\mathrm{J}}(x) = (1-x)^\alpha (1-x)^\beta$ ($\alpha > -1$, $\beta > -1$), $\rho_{\mathrm{B}}(x) = x^\alpha \mathrm{e}^{-2/x}$ ($\alpha \neq -2, -3, \ldots$), $\rho_{\mathrm{L}}(x) = x^\alpha \mathrm{e}^{-x}$ ($\alpha > -1$), $\rho_{\mathrm{H}}(x) = \mathrm{e}^{-x^2}$. Finally, the eigenvalue $\lambda_n$ is computed from $2\lambda_n = -n[2\tau' + (n-1)\sigma'']$.

The $\{Q_n(x)\}_{n\in\mathbb{N}}$ family, also classical, verify the relations (2), (3) and (4), called ($\bar{2}$), ($\bar{3}$) and ($\bar{4}$) where the corresponding constants or polynomials will be denoted with an upper bar: $(\bar{A}_n, \bar{B}_n, \bar{C}_n)$, $(\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n)$, $(\bar{\sigma}, \bar{\tau}, \bar{\lambda}_n)$.

The link between the $P_n$ and the $Q_m$ given by (1) can easily be replaced by a linear relation involving only the $Q_m$ using (4):

$$\sigma \sum_{m=0}^{n} [C_m(n) Q_m''] + \tau \sum_{m=0}^{n} [C_m(n) Q_m'] + \lambda_n \sum_{m=0}^{n} [C_m(n) Q_m] = 0. \tag{5}$$

Notice that from (5) a linear system can be readily obtained by equating to zero the coefficients of $x^k$, $k = 0, 1, \ldots, n$. However, the coefficient of $x^n$ is

$$C_n(n) \left\{ \frac{n(n-1)}{2} \sigma'' + n\tau' + \lambda_n \right\}$$

which is identically zero ($\equiv 0$) (see the aforementioned expression for the eigenvalue $\lambda_n$). So, this linear system has $n+1$ unknowns (the $C_m(n)$ coefficients) and only $n$ equations. Of course, an equivalent linear system could also be obtained by expanding the polynomials $\sigma Q_m''$ ($m = 2, \ldots, n$) and $\tau Q_m'$ ($m = 1, \ldots, n$) in the $Q_m$ basis, which should also have $n+1$ unknowns and only $n$ linearly independent equations at most.

However, these two linear systems are not very useful because they strongly depend on the coefficients of the polynomials $Q_m$ ($m = 0, \ldots, n$). Instead of using them, let us proceed as follows:

Multiplication of (5) by $\bar{\sigma}$ and use of ($\bar{4}$) and ($\bar{3}$) gives

$$\begin{aligned} &\sum_{m=0}^{n} C_m(n) \sigma [-\bar{\tau} Q_m' - \bar{\lambda}_m Q_m] \\ &\quad + \sum_{m=0}^{n} C_m(n) \tau [\bar{\alpha}_m Q_{m+1} + \bar{\beta}_m Q_m + \bar{\gamma}_m Q_{m-1}] + \lambda_n \left[ \sum_{m=0}^{n} C_m(n) \bar{\sigma} Q_m \right] = 0 \end{aligned} \tag{6}$$

where, as pointed out in (2), $Q_{-1} = 0$. Since multiplication of (5) by a polynomial does not increase the number of linearly independent equations (at most $n$ in (5)), it turns out that (6) gives rise to a linear system which contains at most $n$ linearly independent equations.

Next, multiplication of (6) by $\bar{\sigma}$ together with the use of ($\bar{3}$) allows to eliminate in (6) the term depending on $Q'_m$. The last step consists to expand the remaining terms of type $\sigma\bar{\tau}Q_k$ $(k = 1, \dots, n+1)$, $\bar{\sigma}^2 Q_k$ $(k = 0, \dots, n)$, $\sigma\bar{\sigma}Q_k$ $(k = 2, \dots, n)$ and $\tau\bar{\sigma}Q_k$ $(k = 0, 1, \dots, n+1)$ in linear combination of $Q_m$ (with constants coefficients) by using ($\bar{2}$) repetitively.

After this process (6) reduces to

$$\sum_{m=0}^{N} M_m[C_0(n), \dots, C_n(n)]\,Q_m(x) = 0 \tag{7}$$

where the notation

$$N = \max\{n + \deg(\sigma) + \deg(\bar{\sigma}),\ n + 2\deg(\bar{\sigma}),$$
$$n + 1 + \deg(\tau) + \deg(\bar{\sigma}),\ n + 1 + \deg(\sigma) + \deg(\bar{\tau})\} \quad (N \geqslant n+1)$$

has been used.

Finally, from (7) we deduce the linear system we are looking for:

$$M_m[C_0(n), \dots, C_n(n)] = 0, \quad m = 0, 1, \dots, N. \tag{8}$$

It is easy to prove that the range of its coefficient matrix is greater than or equal to $n$. On the other other hand, since (7) is obtained from (6) by multiplying the latter equation by a polynomial, the number of linearly independent equations in (8) has to be the same as in (6), i.e. $n$ at most. So, choosing in (8) $n$ equations, the connection coefficients $C_m(n)$ can be obtained in terms of one of them which is arbitrary. Clearly, the remaining equations $(N - n > 0)$ are then satisfied identically.

The relations (8) contains (linearly) several $C_i(n)$ depending essentially on the degrees of $\sigma$ and $\bar{\sigma}$. In the most complicated situations: connection between two Jacobi, between Jacobi and Bessel or between Bessel and Jacobi, $\sigma$ and $\bar{\sigma}$ are polynomials of second degree and the structure relation (3) contains effectively three terms; so expansions of type $\bar{\sigma}^2 Q_m$ mixes already $Q_{m+4}, Q_{m+3}, \dots, Q_{m-4}$. In these three situations the relations (6) link (in general and for $n \geqslant 8$) from $Q_{m+4}$ to $Q_{m-4}$ which means that the collecting polynomials $Q_m$ in (7) give a relation of type (maximum)

$$M_m[C_{m+4}, \dots, C_{m-4}] = 0 \tag{9}$$

which is valid for $n$ greater than or equal to the number of initial conditions needed to start the recursion ($n \geqslant 8$ in this case). Notice that when $n$ is less than the number of initial conditions, system (8) also gives the solution, but not in a recurrent way.

The starting values of the recurrence (9) can be computed from the linear system (8). Choosing $C_n(n)$ as the arbitrary coefficient, the simplest way is to consider as initial conditions $C_n(n), C_{n-1}(n), \dots, C_{n-7}(n)$, where $C_i(n)$ $(i = n-7, \dots, n-1)$ are determined from (8) in terms of $C_n(n)$. This is always possible due to the lower triangular structure of the corresponding coefficients matrix. Then, $C_n(n)$, which only depends on the relative normalization of the two families $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$, can be easily obtained by identification of the highest power in the expansion (1).

## 2. Extensions

The approach we have just described can also be applied in more general situations. For *semi-classical* orthogonal $P_n$ and $Q_n$, (3) should be replaced by

$$\sigma P_n' = \sum_{k=n-s-1}^{n+t-1} \beta_{k,n} P_k \tag{10}$$

and $(\bar{3})$ by the analogous formula $(\overline{10})$. In the latter equation, $t$ is the degree (which can be arbitrary in the semi-classical class) of the polynomial $\sigma := \sigma(x)$ and $s$ is an integer characterizing the class. Moreover, (4) and $(\bar{4})$ are still valid for the semi-classical families except that the coefficients of $P_n''$, $P_n'$ and $P_n$ (respectively $Q_n$) are now polynomials of fixed degree but with coefficients depending on $n$.

If $P_n$ belongs to the *Laguerre–Hahn* class and $Q_n$ is *semi-classical* this technique also works. In this case, $P_n$ is solution of a fourth-order differential equation replacing (4). But to save computations it is possible to use equivalently a differential relation [7] which is very simple for instance for the associated polynomials $P_{n-1}^{(1)}(x)$ of the $P_n(x)$ ($P_n$ being classical):

$$L_2^*[P_{n-1}^{(1)}(x)] = (\sigma'' - 2\tau')P_n'(x)$$

with $L_2^*$ the adjoint operator of $L_2$:

$$L_2^*[z(x)] = [\sigma z(x)]'' - [\tau z(x)]' + \lambda_n z(x)$$

and

$$P_{n-1}^{(1)}(x) = \frac{1}{c_0}\int_I \frac{P_n(x) - P_n(s)}{x - s}\rho(s)\,\mathrm{d}s \quad \text{with } c_0 = \int_I \rho(s)\,\mathrm{d}s, \tag{11}$$

$I$ being the orthogonality interval (the unit circle in the Bessel case).

## 3. Examples

These techniques are very easily implemented in any symbolic language in both cases: semi-classical in semi-classical and Laguerre–Hahn is semi-classical. Just as an example we derive, using Mathematica [11], the recurrence relations for the connection coefficients in the expansions:

$$P_{n-1}^{(1)}(x) = \sum_{m=0}^{n-1} C_m(n-1)P_m(x)$$

where the monic $P_m$ belongs to any classical family, taking care of the initial conditions matching properly the polynomials for $m = n - 1$. For instance, from the definition (11), keeping $P_{n-1}^{(1)}(x)$ also monic, we obtain immediately that $C_{n-1}(n-1) = 1$.

Now we give the results for the Hermite case which coincides with a known result [3] and for the Laguerre and Bessel ($\alpha = 0$) cases, which seems to be new. These coefficients $C_m(n-1)$ are denoted in the Mathematica algorithms by **CA**$[m, n-1]$ (for connection associated).

### 3.1. Connection coefficients between first associated Hermite and Hermite

Let us consider the expansion for monic polynomials:

$$H_{n-1}^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1+n] H_m(x).$$

Then, by using the technique described above, the recurrence relation for the connection coefficients $\mathbf{CA}[m, n-1]$ is a two-term one given by

$$2(1+m+n)\mathbf{CA}[m, -1+n] + 2(1+m)(2+m)\mathbf{CA}[2+m, -1+n] = 0$$

with $m = n-3, n-4, \ldots, 2, 1, 0$ and the initial conditions

$$\mathbf{CA}[-1+n, -1+n] = 1,$$

$$\mathbf{CA}[-2+n, -1+n] = 0.$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 2$.

### 3.2. Connection coefficients between first associated Laguerre and Laguerre

Let us consider the expansion for monic polynomials:

$$[L_{n-1}^{(a)}]^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1+n] L_m^{(a)}(x).$$

Then, by using the technique described above, the recurrence relation for connection coefficients $\mathbf{CA}[m, n-1]$ becomes in this case a four-term one given by

$$\begin{aligned}
&(m+n)\mathbf{CA}[-1+m, -1+n] \\
&\quad + (1+a+3m+am+4m^2+n+an+2mn)\mathbf{CA}[m, -1+n] \\
&\quad + (1+m)(1+a+m)(4+5m+n)\mathbf{CA}[1+m, -1+n] \\
&\quad + 2(1+m)(2+m)(1+a+m)(2+a+m)\mathbf{CA}[2+m, -1+n] = 0,
\end{aligned}$$

where $m = n-3, n-4, \ldots, 2, 1$ and the initial conditions are

$$\mathbf{CA}[-3+n, -1+n] = 8+2a-10n-an+3n^2,$$

$$\mathbf{CA}[-2+n, -1+n] = 2-2n,$$

$$\mathbf{CA}[-1+n, -1+n] = 1.$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 3$.

### 3.3. Connection coefficients between first associated Bessel and Bessel

Let us consider the expansion for monic polynomials:

$$[B_{n-1}^{(0)}]^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1+n] B_m^{(0)}(x).$$

Then, by using the technique described above, the recurrence relation for connection coefficients $\mathbf{CA}[m, n-1]$ is a five-term one given by

$$\begin{aligned}
&(-2+m-n)(-1+m+n)\mathbf{CA}[-2+m, -1+n] \\
&\quad + 4(1-m)\mathbf{CA}[-1+m, -1+n] \\
&\quad + \frac{2(-6+7m+7m^2+n+n^2)}{(-1+2m)(3+2m)}\mathbf{CA}[m, -1+n] \\
&\quad - \frac{4(2+m)}{(1+2m)(3+2m)}\mathbf{CA}[1+m, -1+n] \\
&\quad + \frac{(2+m-n)(3+m+n)}{(1+2m)(3+2m)^2(5+2m)}\mathbf{CA}[2+m, -1+n] = 0,
\end{aligned}$$

where $m = n-3, n-4, \ldots, 2$ and the initial conditions are

$$\begin{aligned}
&\mathbf{CA}[-4+n, -1+n] = \frac{-11+12n-4n^2}{3(-5+2n)(-1+2n)}, \\
&\mathbf{CA}[-3+n, -1+n] = \frac{9-16n+8n^2}{3(-3+2n)(-1+2n)}, \\
&\mathbf{CA}[-2+n, -1+n] = -1, \\
&\mathbf{CA}[-1+n, -1+n] = 1.
\end{aligned}$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 4$.

**Remark.** In a recent work [8] we apply a similar technique to the linearization problems:

$$P_i(x)P_j(x) = \sum_{k=0}^{i+j} L_{i,j,k} P_k(x)$$

in which we examine the complexity of the recurrence relation between linearization coefficients as function of orthogonality restrictions. For the connection coefficients the three levels of assumptions are: without orthogonality in the $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$ families (arbitrary basis), the coefficients $C_m(n)$ are of course also *arbitrary*; with orthogonality in each family $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$, the recurrence relations for the $C_m(n)$ *mixed both indices* (from both recurrence relations satisfied by the $P_n$ and $Q_n$ families after multiplication of (1) by $x$); with semi-classical (classical) assumption, the recurrence relations for the $C_m(n)$ *keeps the same argument* $n$ (from the existence of the structure relation (3)).

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